A Passive 2DOF Walker: Hunting for Gaits using Virtual Holonomic Constraints

Leonid B. Freidovich[1], Uwe Mettin[1], Anton S. Shiriaev[1,2], Mark W. Spong[3]

[1]Department of Applied Physics and Electronics, Umeå University, Sweden
[2]Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, Norway
[3]Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

1 Hybrid Dynamics of the Compass-Gait Biped

We consider the standard model of a two-link passive compass-gait biped robot:

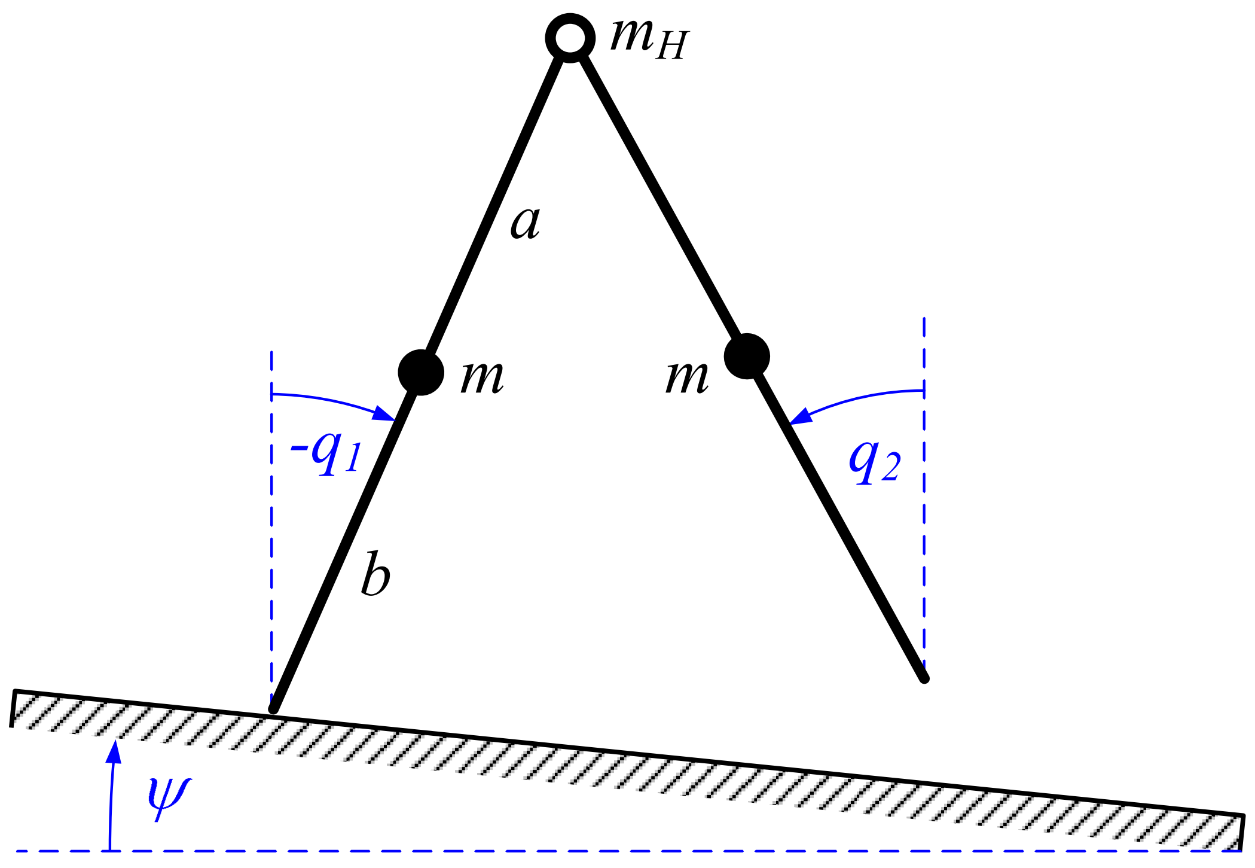


Figure 1: Schematic of the compass-gait biped on a shallow slope ψ. Here, q_1 and q_2 give the angular positions of the stance and swing legs, respectively.

Table 1: Physical parameters of the biped.

Parameter	Legs	Hip
Mass [kg]	$m = 5$	$m_H = 10$
Distance to CoM [m]	$a = b = 0.5$	
Length [m]	$l = a + b$	
Gravitational constant	$g = 9.81$ m/s^2	

The dynamics of the robot can be described by the following system of Euler-Lagrange equations with impulsive effects

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot q)}{\partial \dot q}\right] - \frac{\partial \mathcal{L}(q,\dot q)}{\partial q} = 0,\ q \notin \mathcal{S} \qquad (1)$$
$$q^+ = P\,q^-,\quad \dot q^+ = P_q(q^-)\,\dot q^-,\qquad q^- \in \mathcal{S}$$

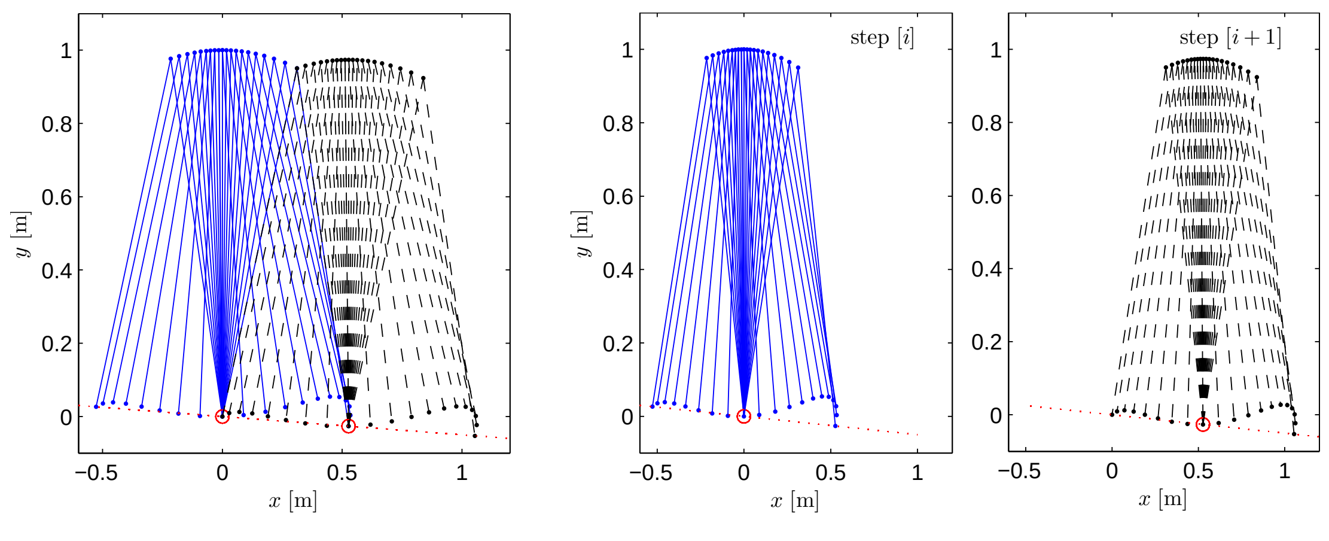
where $q = [q_1,\ q_2]^T$ is the vector of generalized coordinates, we use the notation

$$q^- = \lim_{\tau\to t-} q(\tau) \qquad \text{and} \qquad q^+ = \lim_{\tau\to t+} q(\tau)$$

for the states right before and right after the impact triggered by appropriate crossing of the switching surface

$$\mathcal{S} = \{q:\ \cos(q_1+\psi) - \cos(q_2+\psi) = 0\} \qquad (2)$$

Our goal is to find possible symmetric gaits:



(a) A symmetric gait described by intermediate time instants.

(b) A sequence of two identical steps from the respective legs that form a symmetric gait.

Figure 2: Schematic of a symmetric gait of a passive planar two-link walker that experiences an impact after each step, shown in the Cartesian frame.

2 Direct procedure

Let us introduce the following notation for a periodic trajectory

$$q_\star(0+) = [\,a,\ e\,]^T,\qquad \dot q_\star(0+) = [\,b,\ f\,]^T$$
$$q_\star(T-) = [\,c,\ g\,]^T,\qquad \dot q_\star(T-) = [\,d,\ h\,]^T$$

It can be shown that

$$g = a,\ c = e = -a - 2\psi,\ f = \frac{b\cos(2a+2\psi)p_2 - p_6 d}{p_3}$$

$$h = \frac{d(p_3p_7 - p_6p_2) - b(p_3p_1 - p_2^2\cos(2a+2\psi))}{p_3p_6}\cos(2a+2\psi)$$

So, we have the following optimization problem:

$$\min_{\{a,b,d,T\}}\left\{\left\|\bar q(T) - \begin{bmatrix} c \\ g\end{bmatrix}\right\|^2 + \left\|\dot{\bar q}(T) - \begin{bmatrix} d \\ h\end{bmatrix}\right\|^2 :\ \bar q(0) = [\,a,\ e\,]^T,\quad \dot{\bar q}(0) = [\,b,\ f\,]\right\} \qquad (3)$$

for the solutions of (1) over $[0, T]$.

3 Virtual Holonomic Constraint

Substituting the relations

$$q_1 = \theta,\qquad q_2 = \varphi(\theta) \qquad (4)$$

into the Euler-Lagrange equations and collecting terms, we obtain two scalar differential equations of second order for θ

$$\alpha_1(\theta)\,\frac{d^2}{dt^2}\theta + \beta_1(\theta)\,\left[\frac{d}{dt}\theta\right]^2 + \gamma_1(\theta) = 0 \qquad (5)$$
$$\alpha_2(\theta)\,\frac{d^2}{dt^2}\theta + \beta_2(\theta)\,\left[\frac{d}{dt}\theta\right]^2 + \gamma_2(\theta) = 0 \qquad (6)$$

which are integrable. Moreover, any linear combination of the equations (5), (6) with any θ-dependent weights has again the form

$$\alpha(\theta)\,\ddot\theta + \beta(\theta)\,\dot\theta^2 + \gamma(\theta) = 0 \qquad (7)$$

and it can be shown that the energy function, if well-defined for some constant x,

$$E_x(\theta,\dot\theta) = \frac{1}{2}\underbrace{e^{\left\{\int_x^\theta \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}}}_{\Psi_x(\theta)}\dot\theta^2 + \underbrace{\int_x^\theta \frac{\gamma(\tau)}{\alpha(\tau)}\Psi_x(\tau)d\tau}_{\Pi_x(\theta)} \qquad (8)$$

preserves its value.

In particular, one restore the true energy

$$E(q,\dot q) = \frac{1}{2}\dot q^T M(q)\,\dot q + V(q)$$

of the Euler-Lagrange system provided that the the generalized coordinates satisfy (4)

$$E(q,\dot q)\Big|_{\substack{q_1 = \theta,\ q_2 = \phi(\theta)\\ \dot q_1 = \dot\theta,\ \dot q_2 = \phi'(\theta)\dot\theta}} = E_0(\theta,\dot\theta) \qquad (9)$$
$$= \left(\frac{p_1}{2} - p_2\cos(\theta - \varphi(\theta))\varphi'(\theta) + \frac{p_3}{2}(\varphi'(\theta))^2\right)\dot\theta^2 + p_4(\cos(\theta) - 1) + p_5(1 - \cos(\varphi(\theta)))$$

where the function $E_0(\cdot)$ is from (8) with $x = 0$.

It is useful to observe that

$$\theta_\star(0) = a,\ \dot\theta_\star(0) = b,\ \theta_\star(T) = c,\ \dot\theta_\star(T) = d$$
$$\varphi(a) = e,\ \varphi'(a) = f/b,\ \varphi(c) = g,\ \varphi'(c) = h/d \qquad (10)$$

4 New searching procedure

One can look at the system of differential equations (5) and (6) as a system of algebraic equations for the two unknowns $\dot\theta_\star^2(t)$ and $\ddot\theta_\star(t)$ and obtain

$$\dot\theta_\star^2 = \frac{\alpha_2(\theta_\star)\gamma_1(\theta_\star) - \alpha_1(\theta_\star)\gamma_2(\theta_\star)}{\alpha_1(\theta_\star)\beta_2(\theta_\star) - \alpha_2(\theta_\star)\beta_1(\theta_\star)}$$

which in combination with (9) gives

$$\varphi''(\theta_\star) = f_0(a, b, \theta_\star, \varphi(\theta_\star), \varphi'(\theta_\star)) \qquad (11)$$

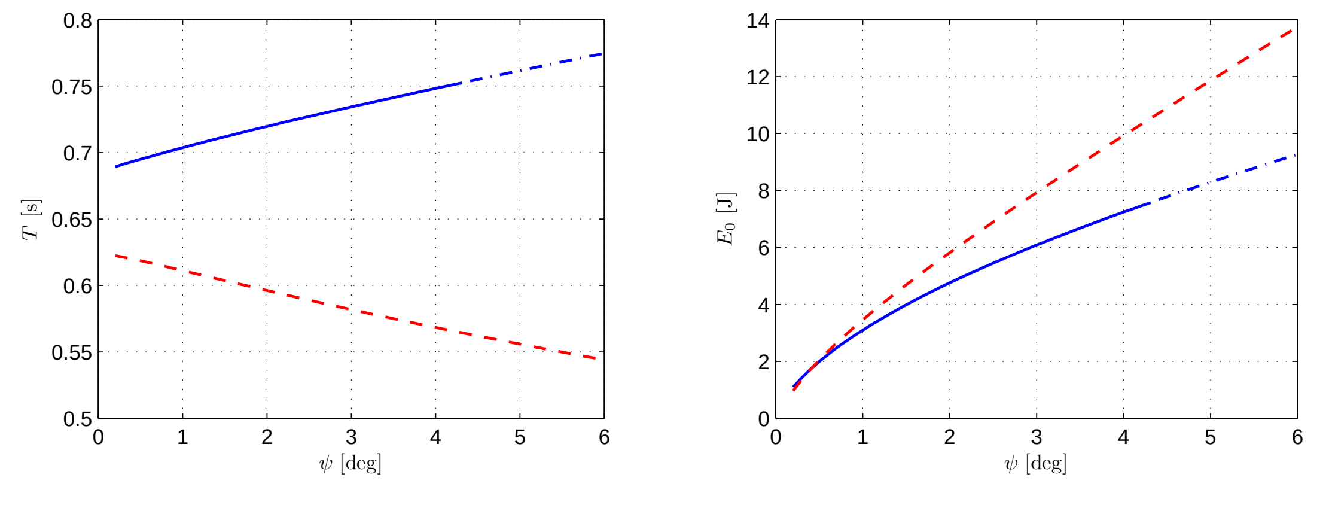
The result is the following optimization problem

$$\min_{\{a,b\}}\left\{|\bar\varphi(c) - g|^2 + |\bar\varphi'(c) - h/d|^2 :\ \bar\varphi(a) = e \text{ and } \bar\varphi'(a) = f/b\right\} \qquad (12)$$

for the solutions of (11) over $[a, c]$.

5 Numerical Results

Two cycles were found using (12) for each value of $\psi \in (0, \sim 6)$ deg:

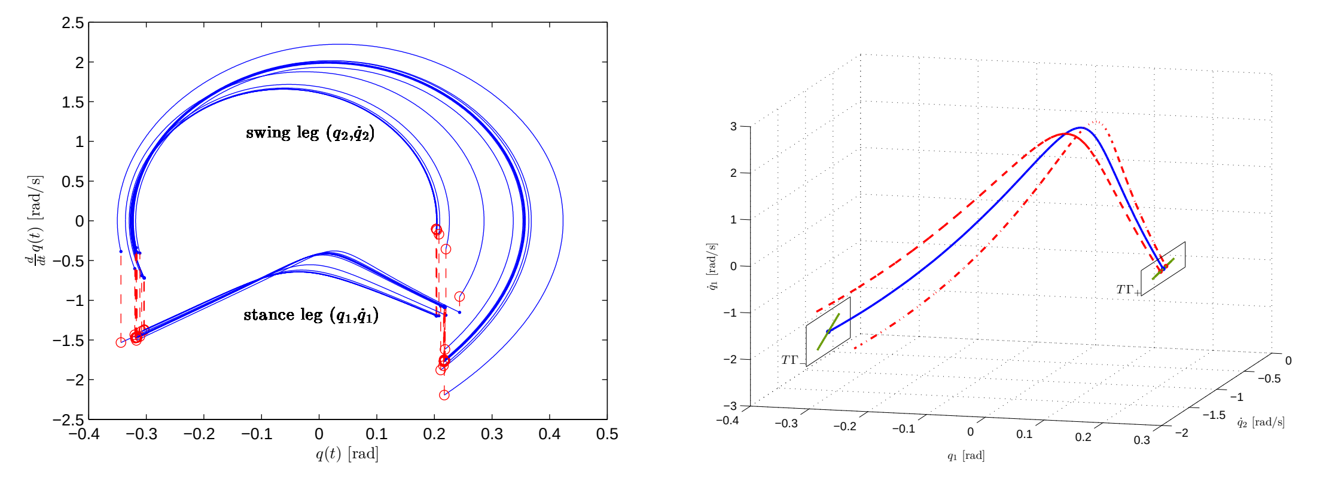


(a) Resulting half-period T.

(b) Required total energy E_0.

Figure 3: Half-period and total energy of the two symmetric gait cycles obtained from analysis. The cycle #2 represented by the dashed line is unstable while the cycle #1 represented by the solid line is exponentially orbitally stable within the interval $\psi \in (0, \sim 4.4)$ deg

The case of $\psi = 2.87$ deg is illustrated below



(a) Divergence from an unstable limit cycle #2 and convergence to a stable cycle #1.

(b) Periodic trajectory in subspace $\{q_1, \dot q_1, \dot q_2\}$. There is no invariant hybrid zero dynamics.

Figure 4: Simulation results for $\psi = 2.87$ deg.