Stabilization of a Walking Motion with Unstable Hybrid Zero Dynamics via an Analytically Computed Transverse Linearization

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1 The Compass Biped Walker

We consider a model of a two-link planar biped robot known as the *compass biped walker*. The angle of the stance leg q_2 is chosen as the independent variable θ, and the angle between the legs q_1 is actuated and should satisfy a **virtual constraint** $y = 0$ defined by $y := q_1 - \phi(q_2)$. where $\phi(\cdot)$ is a fourth-order Bezier polynomial with coefficients deliberately chosen to create a walking cycle with **unstable hybrid zero dynamics**.

This motion is somewhat unusual: the swing leg is held back for most of the continuous phase, and swings forward quickly at the end (See an animation of the trajectory along the bottom of the poster). Such a motion could, for example, be used to kick a football, or a stylized dancing motion for an entertainment robot.

2 The Transverse Linearization

Now, the transverse coordinates of the nonlinear system are given by $x_\perp := \Big[I(\theta,\dot\theta,\theta_\star(0),\dot\theta_\star(0)),\ \ y,\ \ \dot y\Big]^T$.

First step in controller design is to consider the hybrid linear comparison system.

$$\begin{aligned}\dot\zeta(t) &= A(t)\,\zeta(t) + B(t)\,v(t),\\ \zeta(kT)_+ &= d^{TS}F\,\zeta(kT_h)_-, \quad k = 1,2,3,\ldots\end{aligned}$$

where T_h is period of nominal motion. If we set $v = 0$, then we are considering the case with just the feedback-linearizing control signal. We can calculate the state transition matrix over the continuous cycle, denoted $\Phi(0,T)$. Then a linearization of the **Poincaré map** is given by:

$$dF^{TS}\ \Phi(0,T_h) = \begin{bmatrix}1.2 & 0.18 & 0.13\\ 0 & 0.1 & 0.04\\ 0 & -2.1 & -0.79\end{bmatrix}.$$

If the virtual constraint is satisfied ($y = \dot y = 0$) then the first coordinate I grows each period by 20%.

Therefore, a control design which simply keeps the virtual constraint satisfied **cannot stabilize this walking cycle**. See Fig. 1 (Left).

3 Stabilization via Transverse Linearization

Then we construct a state-feedback controller of the following form:

$$u(y,\theta,\dot y,\dot\theta) = N(y,\theta)^{-1}[K(\theta)z(y,\theta,\dot y,\dot\theta) - R(y,\theta,\dot y,\dot\theta)], \tag{1}$$

$$K(\theta) = -R_c^{-1}(s)B(s)^TP(s), \quad s = \Theta^{-1}(\theta). \tag{2}$$

where $\Theta^{-1} : [\theta_+,\theta_-] \to [0,T_h]$ is a projection operator and $P(t)$ is the solution of the Riccati differential equation

$$-\dot P(t) = P(t)A(t) + A^T(t)P(t) - P(t)B(t)R_c^{-1}(t)B^T(t)P(t) + Q_c(t), \tag{3}$$

over the interval $[0,T_h]$ with $P(T_h) = d^{TS}F^TW_cd^{TS}F$, for some choice of matrix weighting functions $Q_c(t) \geq 0$, $W_c > 0$, $R_c(t) > 0$.

Proposition 1 If one selects W_c, Q_c, and R_c such that the solution of (3) satisfies $P(0) < W_c$, then the controller (1), (2) applied to the nonlinear system renders the target cycle **exponentially orbitally stable**.

A controller was designed with technique described above, and the weighting matrices $W_c = \text{diag}([5,5,1])$, $Q_c = 1\times10^{-3}I_3$, $R_c = 1\times10^{-4}$. Calculating the resulting closed-loop transition matrix $\Phi_{cl}(0,T)$ gives a linearized Poincaré map with stable eigenvalues of $0.4612,\ 3.7\times10^{-3}$, and -2×10^{-7}. See Fig. 1 (Right).

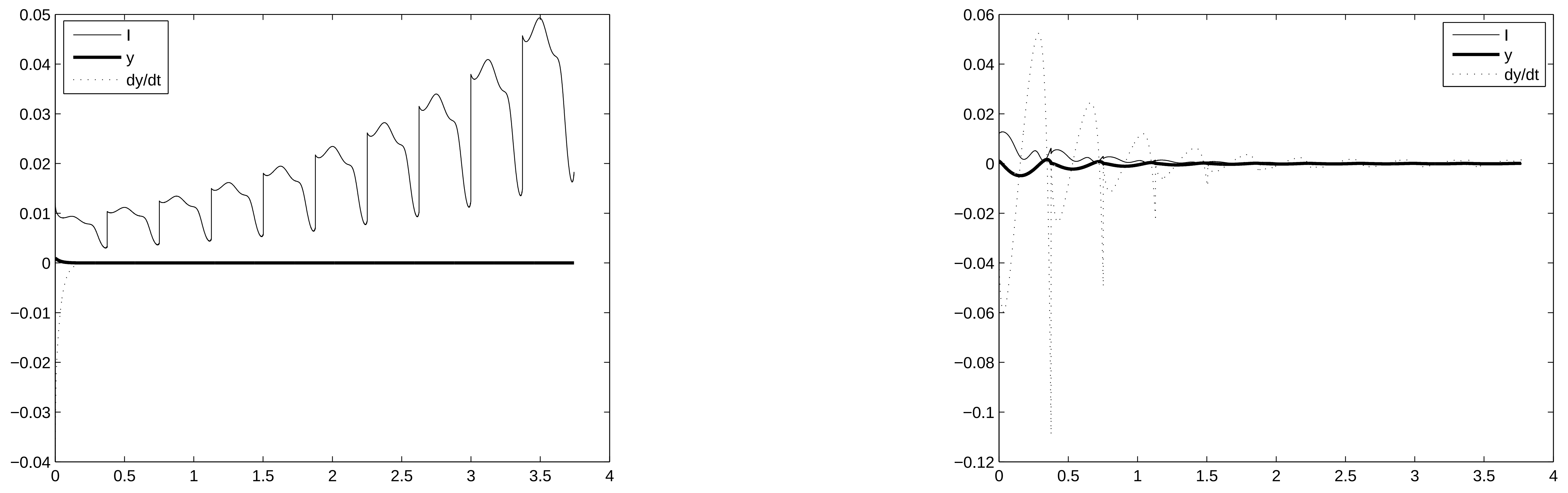

Figure 1: (Left) Unstable behavior when a high-gain PD controller is used to null the virtual constraint; (Right) exponentially stable walking motion using a controller based on the full transverse linearization.